ε-Near Collections

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10 Octobre 2011

Outline

(1^o) Problem: **How to measure the nearness of pictures**,
(2^o) Distance function history,
(3^o) Frechét metric vs. Lowen approach distance,
(4^o) Two forms of neighbourhoods of points,
(5^o) Nearness (apartness) of pictures.

ε vs. $\in$

Hairdressers Conference, Punch, 1869

ε vs. $\in$

Hairdressers Conference, Punch, 1869

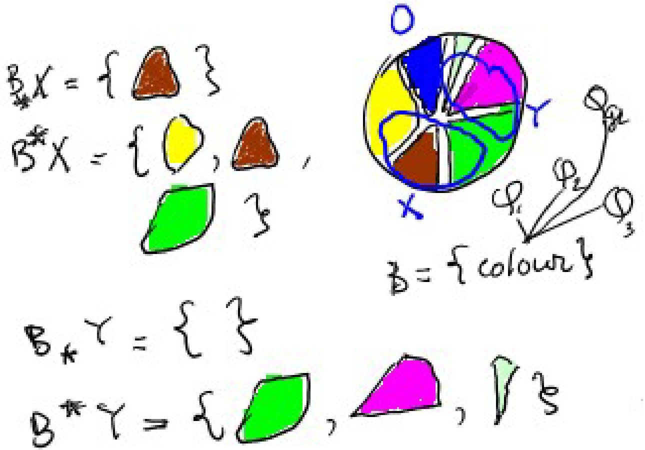
Assume $\varepsilon = 5$.

$$X = \{x : x = \text{mirror reflection}\}.$$

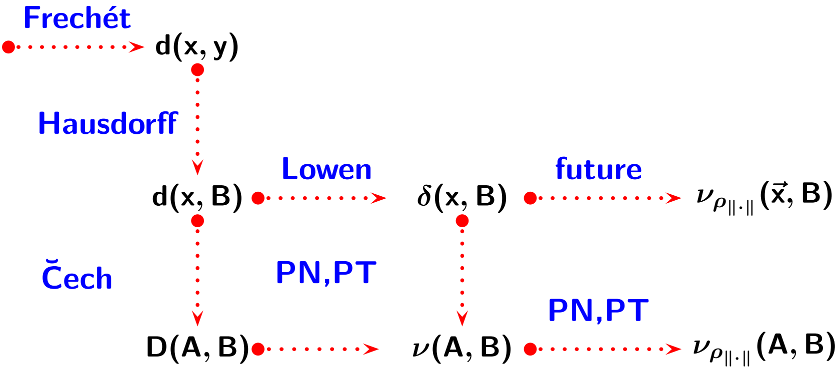
(**Note**.1) **ε is a number.**
(**Note**.2) **# of hairdressers is less than ε.**
(**Note**.3) **Let y = poodle reflection, i.e., $y \in X$.**
(**Note**.4) **$\in$ is a relation.**
(**Note**.5) Paul Erdös referred to children as epsilons.

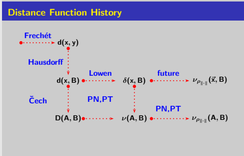
Perceptual Nearness

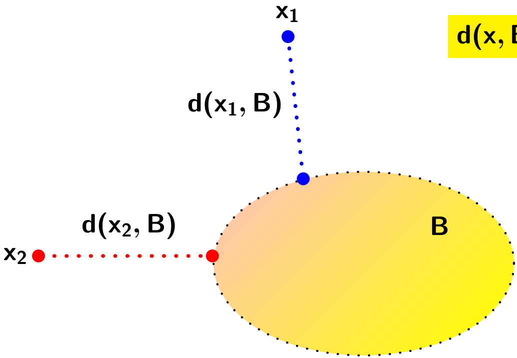
Near Rough Sets



Distance Function History





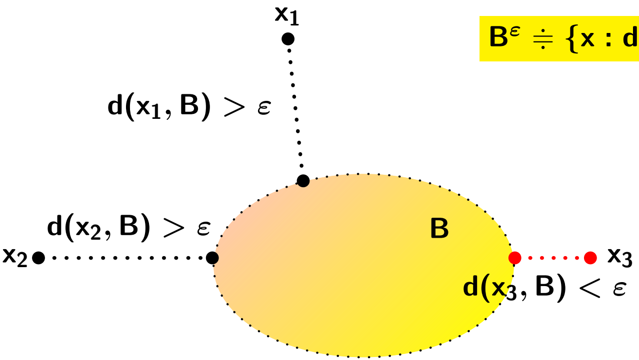


$$d(x, B) \doteqdot \inf\{d(x, b) : b \in B\}.$$

Lowen Boundary Set

$$\varepsilon \in (0, \infty],$$

$$\mathbf{B}^{\varepsilon} \doteq \{x : d(x, B) \leq \varepsilon\}.$$



Frechét Metric vs. Lowen Approach Distance

$d : \mathbf{X} \times \mathbf{X} \rightarrow [0, \infty)$, $\forall x, y, z \in X$, [1]	$\delta : \mathbf{X} \times \boldsymbol{\mathcal{P}}(\mathbf{X}) \rightarrow [0, \infty]$, $\forall A, B, C \in \mathcal{P}(X)$, [4]
(M.1) $d(x, y) \geq 0$, (M.2) $d(x, y) = 0 \iff x = y$, (M.3) $d(x, y) = d(y, x)$, (**M.4**) $d(x, z) \leq d(x, y) + d(y, z)$.	(A.1) $\delta(x, \{x\}) = 0, \forall x \in X$, (A.2) $\delta(x, \emptyset) = \infty$, (A.3) $\delta(x, B \cup C) = \min\{\delta(x, B), \delta(x, C)\}$, (**A.4**) $\delta(A, B) \leq \boldsymbol{\delta(A, B^{(\varepsilon)})} + \varepsilon$.

Frechét Metric vs. Lowen Approach Distance

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Frechét Metric d vs. Lowen Approach Distance δ

(**Diff**.1) Metric **d** point-to-point based vs. Distance δ point-to-set based,

(**Diff**.2) Metric triangle inequality vs. $\boldsymbol{\delta(x, B)}$ boundary set,

(**Diff**.3) Metric usually not extended vs. δ always extended.

Description-Based Hausdorff Distance

$$\vec{x} = (f_1, \ldots, f_i, \ldots, f_n) \in \mathbb{R}^n \textbf{ (feature vector)},$$
$$\vec{b} = (b_1, \ldots, b_i, \ldots, b_n) \in \mathbb{R}^n \textbf{ (feature vector)},$$
$$\rho_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty], \text{ defined by}$$
$$\rho_{\|\cdot\|}(\vec{x}, \vec{b}) = \| \vec{x} - \vec{a} \|_1 = \sum_{i=1}^{n} |x_i - b_i|,$$
$$d(\vec{x}, B) = \inf \{\rho_{\|\cdot\|}(\vec{x}, \vec{b}) : \vec{b} \in \mathbb{R}^n, b \in B\}.$$

Description-Based Hausdorff Distance

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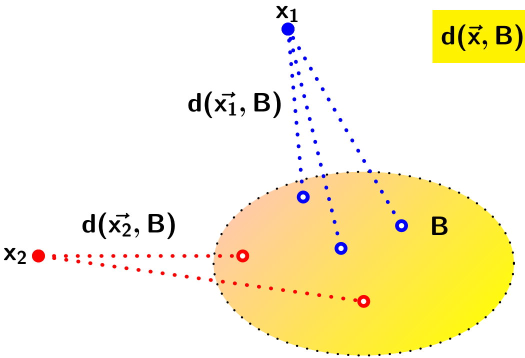
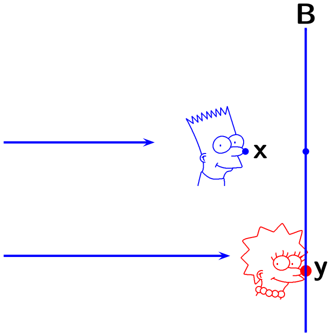


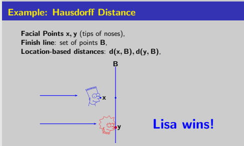
$$\mathbf{d}(\vec{\mathbf{x}}, \mathbf{B}) \doteq \inf \{\rho_{\|\cdot\|}(\vec{\mathbf{x}}, \vec{\mathbf{b}}) : \mathbf{b} \in \mathbf{B}\}.$$

Example: Hausdorff Distance

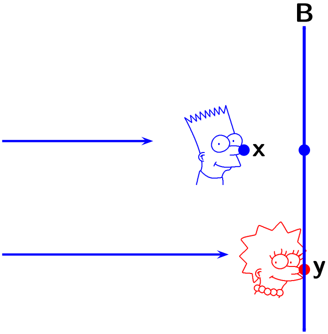
Facial Points x, y (tips of noses),
Finish line: set of points **B**,
Location-based distances: $\mathbf{d(x, B), d(y, B)}$,



Lisa wins!



Facial Point Descriptions $\vec{x}, \vec{y}$ (nose tip descriptions),
Finish line: set of point descriptions $\mathbf{B}$,
Description-based distances: $\mathbf{d(\vec{x}, B), d(\vec{y}, B)}$,



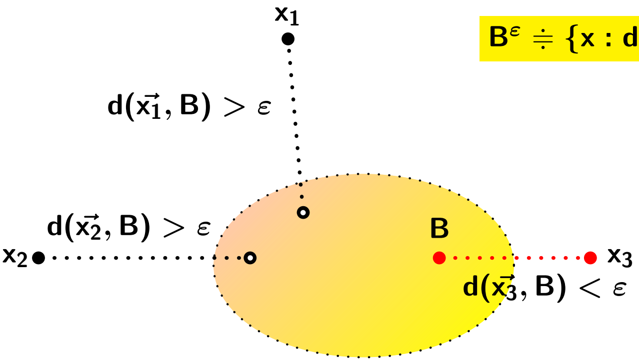
Bart wins!

$\mathbf{d(\vec{x}, B) < d(\vec{y}, B)}$

Description-Based Boundary Set

$$\varepsilon \in (0, \infty],$$

$$B^{\varepsilon} \doteqdot \{x : d(\vec{x}, B) \leq \varepsilon\}.$$



Čech Distance

Let $\mathbf{A}, \mathbf{B} \in \mathcal{P}(\mathbf{X})$.

Čech.1 **Location-based Čech distance**:

$$D(A, B) \doteq \begin{cases} \inf\{d(a, b) : a \in A, b \in B\}, & \text{if } A, B \neq \emptyset, \\ \infty, & \text{if } A \text{ or } B = \emptyset. \end{cases}$$

Čech.2 **Description-based Čech distance**:

$$D_{\rho_{\|\cdot\|}}(A, B) \doteq \begin{cases} \inf\{\rho_{\|\cdot\|}(\vec{a}, \vec{b}) : a \in A, b \in B\}, & \text{if } A, B \neq \emptyset, \\ \infty, & \text{if } A \text{ or } B = \emptyset. \end{cases}$$

Čech Distance

Let $\mathbf{A}, \mathbf{B} \in \mathcal{P}(\mathbf{X})$.

Čech.1 **Location-based Čech distance**:

$$D(A, B) \doteqdot \begin{cases} \inf\{d(a, b) : a \in A, b \in B\}, & \text{if } A, B \neq \emptyset, \\ \infty, & \text{if } A \text{ or } B = \emptyset. \end{cases}$$

Čech.2 **Description-based Čech distance**:

$$D_{\rho_{\|\cdot\|}}(A, B) \doteqdot \begin{cases} \inf\{\rho_{\|\cdot\|}(\vec{a}, \vec{b}) : a \in A, b \in B\}, & \text{if } A, B \neq \emptyset, \\ \infty, & \text{if } A \text{ or } B = \emptyset. \end{cases}$$

Ex.1 **PN Location-based Approach Space**: $(\mathbf{X}, \mathbf{D})$.

Ex.2 **PN Description-based Approach Space**: $(\mathbf{X}, \mathbf{D}_{\rho_{\|\cdot\|}})$.

Ex.3 $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(\mathbf{X})$, **PT Approach Space** $(\mathbf{X}, \nu_{\mathbf{D}})$:

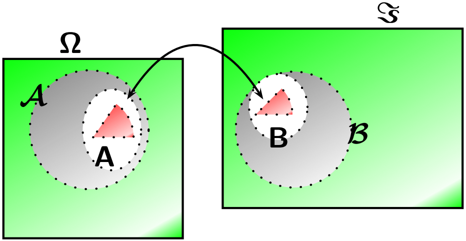
$$\nu_D(\mathcal{A}, \mathcal{B}) \doteqdot \begin{cases} \inf\{\sup_{A \in \mathcal{A}, B \in \mathcal{B}} \{D(A, B) : A \in \mathcal{A}, B \in \mathcal{B}\}\}, & \text{if } \mathcal{A}, \mathcal{B} \neq \emptyset, \\ \infty, & \text{if } \mathcal{A} \text{ or } \mathcal{B} = \emptyset. \end{cases}$$

Ex.4 $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(\mathbf{X})$, **PT Approach Space** $(\mathbf{X}, \nu_{\mathbf{D}_{\rho_{\|\cdot\|}}})$:

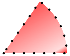
$$\nu_{D_{\rho_{\|\cdot\|}}}(\mathcal{A}, \mathcal{B}) \doteqdot \begin{cases} \inf\{\sup_{A \in \mathcal{A}, B \in \mathcal{B}} \{D_{\rho_{\|\cdot\|}}(A, B) : \dots\}\}, & \text{if } \mathcal{A}, \mathcal{B} \neq \emptyset, \\ \infty, & \text{if } \mathcal{A} \text{ or } \mathcal{B} = \emptyset. \end{cases}$$

Near Pictures

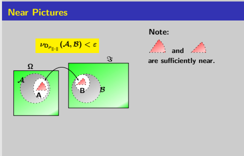
$$\nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathcal{A}, \mathcal{B}) < \varepsilon$$



Note:

 **and** 

are sufficiently near.



Assume $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}, B = \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix}$

$$
\begin{aligned}
\mathcal{A} &= \{A_1, A_2\}, \mathcal{B} = \{B_1, B_2, B_3\}, \text{ where} \\
A_1 &= \{1,2\}, A_2 = \{3,4\}, \\
B_1 &= \{5,6\}, B_2 = \{7,8\}, B_3 = \{9,10\} \\
D(B_1, A_1) &= \inf\{\inf\{d(5,1), d(5,2)\}, \\
&\qquad \inf\{d(6,1), d(6,2)\}\}, \\
&= \inf\{\inf\{4,3\}, \inf\{5,4\}\}, \\
&= \inf\{3,4\} = 3, \dots, \\
\sup(D(B_1, A)) &= \sup\{D(B_1, A_1), D(B_1, A_2)\}, \\
&= \sup\{3,1\} = 3, \dots, \\
\nu(\mathcal{B}, \mathcal{A}) &= \inf\{\sup_{A\in\mathcal{A}}(D(B_1, A), \sup_{A\in\mathcal{A}}(D(B_2, A), \sup_{A\in\mathcal{A}}(D(B_3, A)\}, \\
&= \inf\{3,5,7\} = 3.
\end{aligned}
$$

Two Forms of Neighbourhoods of Points

(**Nbd**.1) **Spherical Neighbourhood**: Let $x \in X$, $\varepsilon \in (0, \infty]$,

$$\mathbf{N_x = \{y \in X : d(x, y) < \varepsilon\}}.$$

(**Nbd**.2) **Description-Based Neighbourhood**:

$$\phi : X \to [0, \infty] \text{ \textbf{probe function}},$$

$$d(\phi(x), \phi(y)) = |\phi(x) - \phi(y)|,$$

$$\mathbf{N_{\phi(x)} = \{y \in X : d(\phi(x), \phi(y)) < \varepsilon\}}.$$

Sufficiently Near Neighbourhoods

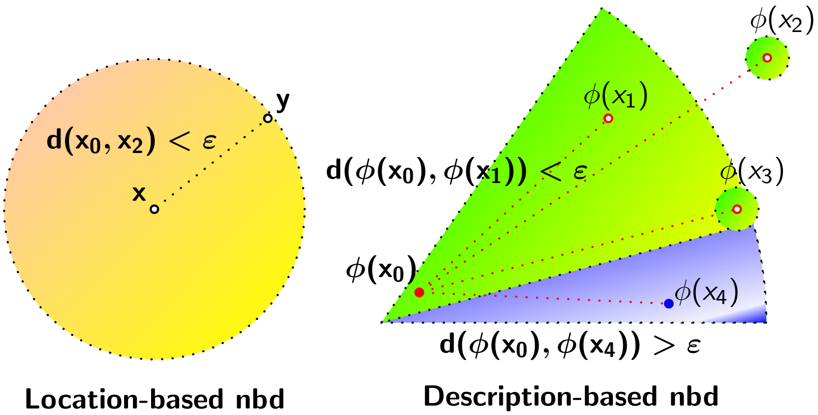
- *Let* $\mathbf{N_x, N_y}$ *denote neighbourhoods of points* x, y, *respectively, and let* $\varepsilon \in \mathbf{(0, \infty]}$.
- **Neighbourhoods are sufficiently near**, *if, and only if,* $\mathbf{N_x, N_y}$ *are close enough,* i.e., $\mathbf{D(N_x, N_y)} < \varepsilon$.

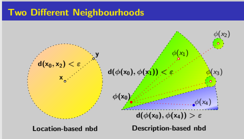
Sufficiently Near Neighbourhoods

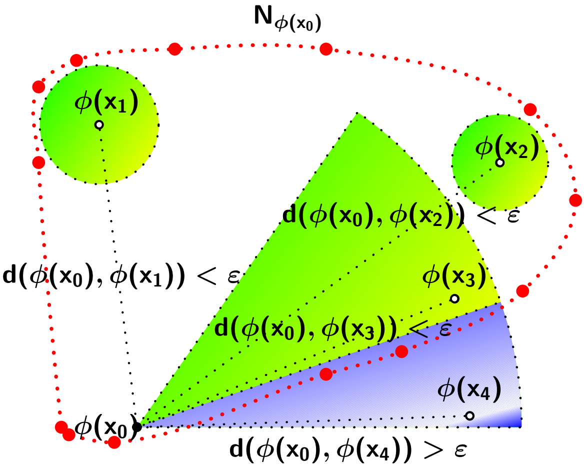
- *Let* $\mathbf{N_x}, \mathbf{N_y}$ *denote neighbourhoods of points* x, y*, respectively, and let* $\varepsilon \in (0, \infty]$.
- **Neighbourhoods are sufficiently near**, *if, and only if,* $\mathbf{N_x}, \mathbf{N_y}$ *are close enough,* i.e., $\mathbf{D(N_x, N_y)} < \varepsilon$.

Neighbourhoods are descriptively sufficiently near, **if, and only if**, $\mathbf{N_{\phi(x)}}, \mathbf{N_{\phi(y)}}$ **are close enough, i.e.**, $\mathbf{D(N_{\phi(x)}, N_{\phi(y)})} < \varepsilon$.

Two Different Neighbourhoods

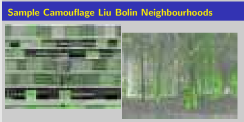



Two Different Neighbourhoods
$d(x_0, x_2) < \varepsilon$
Location-based nbd
Description-based nbd


$N_{\phi(x_0)}$
$\phi(x_1)$
$\phi(x_2)$
$d(\phi(x_0), \phi(x_2)) < \varepsilon$
$d(\phi(x_0), \phi(x_1)) < \varepsilon$
$\phi(x_3)$
$d(\phi(x_0), \phi(x_3)) < \varepsilon$
$\phi(x_4)$
$\phi(x_0)$
$d(\phi(x_0), \phi(x_4)) > \varepsilon$

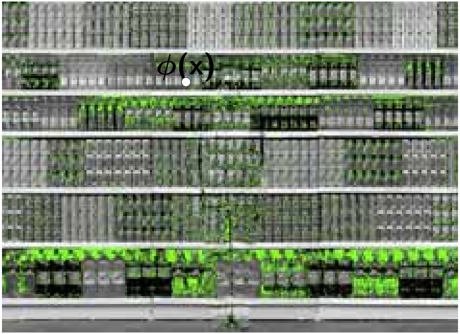
Camouflaged Liu Bolin Sets

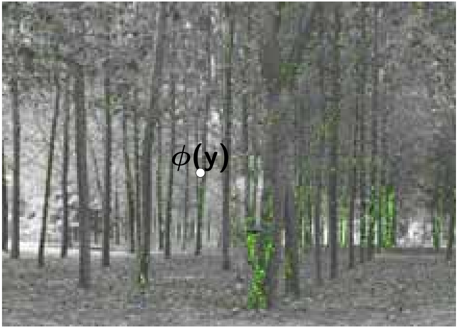
Sample Camouflage Liu Bolin Neighbourhoods

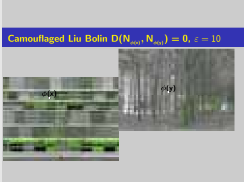


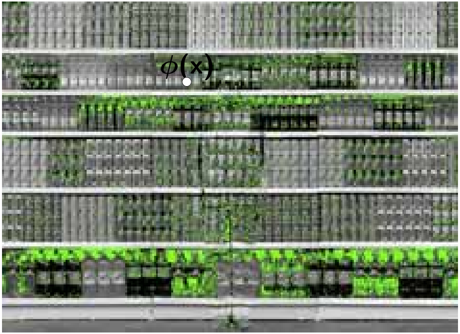
$$
\begin{aligned}
&(\textbf{dist}.1)\ \ \varepsilon = \mathbf{15}, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathsf{N}_{\phi(\mathsf{x})}, \mathsf{N}_{\phi(\mathsf{y})}) = \mathbf{5},\\
&(\textbf{dist}.2)\ \ \varepsilon = \mathbf{20}, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathsf{N}_{\phi(\mathsf{x})}, \mathsf{N}_{\phi(\mathsf{y})}) = \mathbf{3},\\
&(\textbf{dist}.3)\ \ \varepsilon = \mathbf{22}, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathcal{B}, \mathcal{A}) = \mathbf{0}, \text{ for all neighbourhoods}.
\end{aligned}
$$

Camouflaged Liu Bolin $D(N_{\phi(x)}, N_{\phi(y)}) = 0$, $\varepsilon = 10$





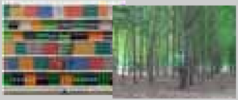




$\varepsilon = 1$

$D(N_{\phi(x)}, N_{\phi(y)}) = 50$

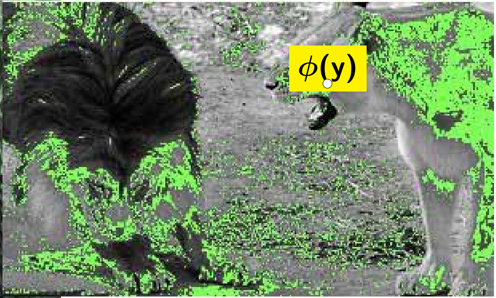
Camouflage ROIs Distances



$$
\begin{aligned}
&(\textbf{dist}.1)\ \ \varepsilon = \mathbf{55}, \nu_{\mathbf{D}_{\rho_{\|\cdot\|}}}(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{A}}) = \mathbf{5},\\
&(\textbf{dist}.2)\ \ \varepsilon = \mathbf{56}, \nu_{\mathbf{D}_{\rho_{\|\cdot\|}}}(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{A}}) = \mathbf{3},\\
&(\textbf{dist}.3)\ \ \varepsilon = \mathbf{57}, \nu_{\mathbf{D}_{\rho_{\|\cdot\|}}}(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{A}}) = \mathbf{0}, \textbf{ for all neighbourhoods in ROIs}.
\end{aligned}
$$

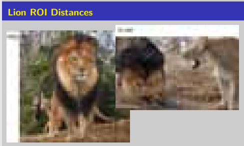
Lion Family

Lion Nbd Distances



$$\begin{aligned}
&(\textbf{dist}.1)\quad \varepsilon = 15, \mathsf{D}(\mathsf{N}_{\rho(\mathsf{x})}, \mathsf{N}_{\phi(\mathsf{y})}) = 8,\\
&(\textbf{dist}.2)\quad \varepsilon = 20, \mathsf{D}(\mathsf{N}_{\rho(\mathsf{x})}, \mathsf{N}_{\phi(\mathsf{y})}) = 0,\\
&(\textbf{dist}.3)\quad \varepsilon = 20, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathcal{B}, \mathcal{A}) = 0, \text{ for all neighbourhoods}.
\end{aligned}$$

Lion ROI Distances



$$
\begin{aligned}
&(\mathbf{dist}.1)\ \ \boldsymbol{\varepsilon = 62, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathcal{B}, \mathcal{A}) = 6},\\
&(\mathbf{dist}.2)\ \ \boldsymbol{\varepsilon = 63, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathcal{B}, \mathcal{A}) = 4},\\
&(\mathbf{dist}.3)\ \ \boldsymbol{\varepsilon = 64, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathcal{B}, \mathcal{A}) = 2},\\
&(\mathbf{dist}.4)\ \ \boldsymbol{\varepsilon = 65, \nu_{\mathsf{D}_{\rho_{\|\cdot\|}}}(\mathcal{B}, \mathcal{A}) = 0}, \textbf{ for all neighbourhoods in ROIs}.
\end{aligned}
$$

Summary: Two Types of Distance

- Location-based Distance between points [1, 2, 3],[8],
- Descripton-based distance between points [5-7,10,11],
- Metrics useful in solving elementary distance problems [2],
- Approach distance functions useful, generally [5-7,11].

Related Works 1

- [1] M. Fréchet,Sur quelques points du calcul fonctionnel, Rend. Circ. Mat. Palermo 22 (1906), 1-74.
- [2] F. Hausdorff. Set Theory. AMS, 1957 (**40 yrs later**).
- [3] E. Čech, Topological Spaces, revised ed. by Z. Frolik, M. Katětov, Wiley Interscience, London, 1966 (**30 yrs later**).
- [4] R. Lowen. Approach Spaces, Clarendon Press, Ox, 1997.

Related Works 2

- [5] S. Pal, J.F. Peters. Rough Fuzzy Image Analysis. Foundations and Methodologies, CRC Press, London, 2010.
- [6] J.F. Peters, S. Tiwari, Approach merotopies & near filters. Theory and Application, Gen.Math.Notes 3 (1), 2011, 14 pp.
- [7] J.F. Peters, S. Naimpally, Approach spaces for near families. Gen.Math.Notes 2 (1), 2011, 1-6.
- [8] S. Tiwari, Some Aspects of Gen. Topology & Applications, Dept. Math.,U of Allahabad, India, Jan. 2010.

Related Works 3

[9] R. Lowen, M. Sioen, D. Vaughan, Completing quasi-metric spaces–An alternative approach, Houston J. Math. 29 (1), 2003, 113-136.

[10] J.F. Peters, P. Wasilewski. Foundations of near sets. Info. Sci. 79, 2009, 3091-3109.

[11] C.J. Henry, Near Sets: Theory and Application, Ph.D. thesis, supervisor: J.F. Peters, Electrical & Computer Engineering, U of Manitoba, 2010.

John Shore, 1750

John Shore, 1750: A tuning fork is an acoustic resonator. It resonates at a specific constant pitch when set vibrating by striking it against a surface or with an object and emits a pure musical tone. The pitch that a tuning fork generates depends on the length of the two prongs (tines). The fork shape produces a very pure tone. The most common tuning fork sounds the note of $A = 440$ Hz.

$$\mathbf{f} = \frac{1}{2\pi \mathbf{l}^2}\sqrt{\frac{\mathbf{AE}}{\rho}}\ \mathbf{Hz}.$$

E, Young's modulus

ρ, material density